PLURICANONICAL MAPS OF VARIETIES OF ALBANESE FIBER DIMENSION TWO

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Abstract. In this paper we prove that for any smooth projective variety of Albanese fiber dimension two and of general type, the 6-canonical map is birational. And we also show that the 5-canonical map is birational for any such variety with some numerical conditions.

1. Introduction

Let X be a smooth complex projective irregular variety of general type, i.e., variety of general type with $q(X) > 0$. We define the Albanese fiber dimension of X to be $e = \dim X - \dim a(X)$, where $a : X \to \mathrm{Alb}(X)$ is the Albanese map. Recently, the birationality of the n-th pluricanonical map $\varphi_{|nK_X|}$ of X has attracted a lot of attention.

When $e = 0$, i.e., X is of maximal Albanese dimension, it was shown by Chen and Hacon [3, 4] that $\varphi_{|6K_X|}$ is birational. This result was improved by Jiang, Lahoz and Tirabassi [7], showing that $\varphi_{|3K_X|}$ is birational. When $e = 1$ or 2, Chen and Hacon [4] proved that $\varphi_{|(5+e)K_X|}$ is a birational map. Recently, Jiang and the author [8] showed that $\varphi_{|4K_X|}$ is birational if $e = 1$. The main results of this paper are some improvements of the result of Chen and Hacon in the case of Albanese fiber dimension two:

Theorem 1.1. *Let X be a smooth complex projective variety of Albanese fiber dimension two and of general type. Then the 6-canonical map $\varphi_{|6K_X|}$ is birational.*

Theorem 1.2. *Let X be a smooth complex projective variety of Albanese fiber dimension two and of general type. If $\dim V^0(\omega_X) > 0$, then $\varphi_{|5K_X|}$ is birational.*

By Theorem 1.1, we can immediately obtain a recent result of Chen, Chen and Jiang:

Corollary 1.3. [2, Theorem 1.1] *Let V be a smooth complex projective irregular 3-fold of general type. Then $\varphi_{|6K_V|}$ is birational.*

By Theorem 1.2 and Lemma 2.3 in the next section, we easily conclude the following corollary.

Corollary 1.4. *Let X be a smooth complex projective variety of Albanese fiber dimension two and of general type. If $p_g(X) > 0$ and $q(X) > \dim X - 2$,then $\varphi_{|5K_X|}$ is birational.*

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2. Definitions and lemmas

In this section, we recall some notion and useful lemmas.

Definition 2.1. Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety Y.

(1) The i-th cohomological support loci of $\mathcal{F}$ is
$$V^i(\mathcal{F}) = \{\alpha \in \mathrm{Pic}^0(Y) | h^i(\mathcal{F} \otimes \alpha) > 0\}.$$
(2) We say $\mathcal{F}$ is IT^0 if $H^i(\mathcal{F} \otimes \alpha) = 0$ for all $i \geq 1$ and all $\alpha \in \mathrm{Pic}^0(Y)$.
(3) $\mathcal{F}$ is called M-regular if $\operatorname{codim} V^i(\mathcal{F}) > i$ for every $i > 0$ and Y is an abelian variety.
(4) We say $\mathcal{F}$ is continuously globally generated at $y \in Y$ (in brief CGG at y) if the natural map
$$\bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes \alpha^\vee \to \mathcal{F} \otimes \mathbb{C}(y)$$
is surjective for any non-empty open subset $U \subset \mathrm{Pic}^0(Y)$.
(5) $\mathcal{F}$ is said to have no essential base point at $y \in Y$ if for any surjective map $\mathcal{F} \to \mathcal{O}_y$, there is a non-empty open subset $U \subset \mathrm{Pic}^0(Y)$ such that for all $\alpha \in U$, the induced map $H^0(\mathcal{F} \otimes \alpha) \to H^0(\mathcal{O}_y \otimes \alpha)$ is surjective.

Lemma 2.2. *Let $\pi : X \to Y$ be a double covering branched along a reduced even divisor $B \in |2L|$, where X is a projective variety, Y is a smooth projective variety and L is a divisor on Y. Let D be a divisor on Y. Then $|\pi^* D|$ induces a birational map if and only if $|\pi^* D|$ separates two general points on two distinct general fibers of π and $H^0(Y, \mathcal{O}_Y(D - L)) \neq 0$.*

Proof. We know that $\pi_* \pi^* \mathcal{O}_Y(D) = \mathcal{O}_Y(D) \oplus \mathcal{O}_Y(D - L)$. Hence we have
$$H^0(X, \mathcal{O}_X(\pi^* D)) = H^0(Y, \mathcal{O}_Y(D)) \oplus H^0(Y, \mathcal{O}_Y(D - L)).$$
The only-if-direction follows from the above equality immediately.

To prove the if-direction, we take an open affine subset $U \subset Y$. Suppose that $U = \operatorname{Spec} R$ for some ring R. We know that $\pi^{-1}(U) = \operatorname{Spec} R[z]/(z^2 - f)$, where $f \in R$ is the local defining equation of B. From the isomorphism of R-modules
$$R[z]/(z^2 - f) \cong R \oplus Rz,$$
it follows that we can choose $s_1, \ldots, s_k, t_1, \ldots, t_n \in R$ such that $1, s_1, \ldots s_k$ is a basis of $H^0(D)$ and $t_1 z, \cdots t_n z$ is a basis of $H^0(D - L)$. Let $y \in Y$ be a general point. Since $H^0(D - L) \neq 0$, we can assume that $f(y) \neq 0$ and $t_1(y) \neq 0$.

Let $\{x_1, x_2\}$ be the preimage of y, where $x_1 = (y, \sqrt{f(y)})$ and $x_2 = (y, -\sqrt{f(y)})$. Since $|\pi^* D|$ separates two general points on two distinct general fibers of π, we can find a section $s_0 + t_0 z \in H^0(X, \mathcal{O}_X(\pi^* D))$ vanishes along y, where $s_0 \in H^0(D)$ and $t_0 z \in H^0(D - L)$. Hence section
$$s_0 + (t_0(y) - t_1(y))\sqrt{f(y)} + t_1 z \in H^0(\pi^* D)$$
vanishes along x_1 but does not vanish along x_2. It follows that $|\pi^* D|$ separates two points on a general fiber of π. Therefore $|\pi^* D|$ induces a birational map. $\square$

Lemma 2.3. *Let X be a smooth complex projective irregular variety. Let F be the general fiber of its the Albanese map. If $p_g(F) > 0$ and $\dim V^0(\omega_X) = 0$, then the Albanese map of X is surjective with connected fibers.*

Proof. The proof is the same as the first step in the proof of [8, Proposition 3,2]. We omit the details and leave it to the reader. □

Lemma 2.4. *Let $f : X \to W$ be a morphism between smooth projective varieties with a general fiber F. Suppose that $\kappa(W) \geq 0$ and K_X is W-big, i.e., $sK_X \geq f^*L$ for some ample divisor L and some integer $s \gg 0$. Suppose further that $h^0(mK_F) > 0$ for some $m \geq 2$. Then after replacing X by an appropriate birational model, there exist positive integers b, c and there is a normal crossing divisor*

$$B \in |bc(m-1)K_X - f^*bM|$$

such that $\lfloor \frac{B}{bc} \rfloor|_F \leq \mathcal{B}_{m,F}$, $\lfloor \frac{B}{bc} \rfloor \leq \mathcal{B}_{m,\alpha}$, for all $\alpha \in \mathrm{Pic}^0(W)$. Here $\mathcal{B}_{m,F}$ (resp. $\mathcal{B}_{m,\alpha}$) is the fixed part of $|mK_F|$ (resp. $|mK_X + f^*\alpha|$), M is a given nef and big divisor on W and b, c are sufficiently large integers depending on M and K_X.

Proof. See [4, Lemma 4.1]. □

Lemma 2.5. *If $\mathcal{F}$ is a coherent sheaf on a smooth projective variety Y and y is a point on Y, then $\mathcal{F}$ is CGG at y if and only if $\mathcal{F}$ has no essential base point at y.*

Proof. Firstly, we assume that $\mathcal{F}$ is CGG at y. For any surjective map $\mathcal{F} \to \mathcal{O}_y$, the induced map $\mathcal{F} \otimes \mathbb{C}(y) \to \mathcal{O}_y$ is also surjective. The definition of CGG implies that the composition

$$\bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes \alpha^\vee \to \mathcal{F} \otimes \mathbb{C}(y) \to \mathcal{O}_y$$

is surjective for any non-empty open subset $U \subset \mathrm{Pic}^0(Y)$. It follows that for any non-empty open subset $U \subset \mathrm{Pic}^0(Y)$, there is an $\alpha \in U$ such that the induced map $H^0(\mathcal{F} \otimes \alpha) \to H^0(\mathcal{O}_y \otimes \alpha)$ is surjective. By semi-continuity, one sees that for a general $\alpha \in \mathrm{Pic}^0(Y)$, the induced map $H^0(\mathcal{F} \otimes \alpha) \to H^0(\mathcal{O}_y \otimes \alpha)$ is surjective. Thus $\mathcal{F}$ has no essential base point at y.

Conversely, suppose that $\mathcal{F}$ has no essential base point at y. One can write $\mathcal{F} \otimes \mathbb{C}(y) = \bigoplus_{i=1}^k V_i$, where $V_i \cong \mathbb{C}(y)$, $i = 1, 2, \ldots, k$. Let $p_i : \mathcal{F} \otimes \mathbb{C}(y) \to V_i$ be the canonical projection and $\varphi_i : \mathcal{F} \to \mathcal{F} \otimes \mathbb{C}(y) \to V_i$ be the composition. Since $\mathcal{F}$ has no essential base point at y, we know that there exists non-empty open subsets U_i $(1 \leq i \leq k)$ of $\mathrm{Pic}^0(Y)$ such that for any $\alpha \in U_i$ the induced map $\tilde{\varphi}_i : H^0(\mathcal{F} \otimes \alpha) \to V_i \otimes \alpha$ is surjective. For any non-empty open subset $U_0 \subset \mathrm{Pic}^0(Y)$, since $\cap_{i=0}^k U_i$ is also a non-empty open subset, the map

$$\bigoplus_{\alpha \in U_0} H^0(\mathcal{F} \otimes \alpha) \otimes \alpha^\vee \longrightarrow \bigoplus_{i=1}^k V_i = \mathcal{F} \otimes \mathbb{C}(y)$$

is surjective. It follows that $\mathcal{F}$ is CGG at y. □

Lemma 2.6. *If $\mathcal{F}$ is a non-zero M-regular sheaf on a complex abelian variety A, then $\mathcal{F}$ has no essential base point at any $y \in A$.*

Proof. The conclusion follows from [10, Proposition 2.13] and Lemma 2.5. □

3. Proof of the main theorems

From now on, we let X be a smooth complex projective variety of Albanese fiber dimension two and of general type. Denote $a : X \to Z = a(X) \subset A$ the Albanese map of X, where A is the the Albanese variety of X. Let $\nu : W \to Z$ be a desingularization of the Stein factorization over Z. Replacing X by an appropriate birational model, we may assume that there is a morphism $f : X \to W$ whose general fiber is a connected smooth surface S. Then we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "a"'] & W \arrow[d, "\nu"] \\ & Z. \end{tikzcd}$$

We will prove this more general statement than Theorem 1.1:

Theorem 3.1. $|6K_X + \alpha|$ *induces a birational map for any* $\alpha \in \mathrm{Pic}^0(X)$.

Proof. By [5, Corollary 2.4], we know that $\varphi_{|6K_X+\alpha|}$ is birational for any $\alpha \in \mathrm{Pic}^0(X)$ if $\varphi_{|4K_S|}$ is birational. Hence we can assume that S satisfies $K_{S_0}^2 = 1$ and $p_g(S) = 2$, where S_0 is the minimal model of S. It is well known that $|4K_S|$ is base point free, and $\varphi_{|4K_S|}$ is a generically double covering onto its image (cf. [1]). Thus the natural morphism $f^*f_*\omega_X^{\otimes 4} \to \omega_X^{\otimes 4}$ define a rational map $X \dashrightarrow Y \subset \mathbb{P}(f_*\omega_X^{\otimes 4})$ over W, where Y is the closure of the image.

Let $Y' \to Y$ be a resolution of singularities of Y, and let $g : X' \to Y'$ be a resolution of indeterminacies of the corresponding rational map $X \dashrightarrow Y'$. We know that g is a generically double covering branched along a reduced even divisor B_1. Let $\mu : \widetilde{Y} \to Y'$ be a log resolution of B_1, such that $B := \mu^*B_1 - 2\lfloor \frac{\mu^*B_1}{2} \rfloor$ is smooth (see [11, Lemma 1.3.1]). Then B is still an even divisor. We assume that $B \in |2L|$ for some divisor L on $\widetilde{Y}$. Let $\widetilde{X} \to \widetilde{Y}$ be the double covering branched along B. One sees that $\widetilde{X}$ is smooth. Thus we obtain $K_{\widetilde{X}} = \pi^*(K_{\widetilde{Y}} + L)$. Now we have the commutative diagram among smooth projective varieties

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\pi"] \arrow[dr, "\widetilde{f}"'] & \widetilde{Y} \arrow[d, "p"] \\ & W. \end{tikzcd}$$

We only need to show that $\varphi_{|6K_{\widetilde{X}}+\alpha|}$ is birational for all $\alpha \in \mathrm{Pic}^0(\widetilde{X})$.

Let H (resp. F) be the fiber of $\widetilde{f}$ (resp. p) over a general point $w \in W$. From the construction, we know that $\pi|_H : H \to F$ is a double covering between smooth surfaces branched along a smooth divisor $B|_F \in |\mathcal{O}_F(2L)|$. Hence we have $K_H = (\pi|_H)^*(K_F + L|_F)$.

By Lemma 2.4, for some $m \geq 2$ and an appropriate birational map $\sigma : \widehat{X} \to \widetilde{X}$ there exist positive integers b, c and there is a normal crossing divisor

$$B_m \in |bc(m-1)K_{\widehat{X}} - g^*bM|$$

such that $\lfloor \frac{B_m}{bc} \rfloor|_{H'} \leq \mathcal{B}_{m,H'}$, $\lfloor \frac{B_m}{bc} \rfloor \leq \mathcal{B}_{m,\alpha}$, for all $\alpha \in \mathrm{Pic}^0(\widehat{X})$. Here M is a given nef and big divisor on W, g is the composite map $\widetilde{f} \circ \sigma$,H' is the general fiber of

g and b, c are sufficiently large integers depending on M and $K_{\widehat{X}}$. Thus we obtain

$$(m-1)K_{\widehat{X}} - \lfloor \frac{B_m}{bc} \rfloor \equiv \frac{1}{c} g^* M + \{\frac{B_m}{bc}\}.$$

By [9, Theorem 10.15], we know that

$$H^i(A, \nu_* g_* \mathcal{O}_{\widehat{X}}(mK_{\widehat{X}} - \lfloor \frac{B_m}{bc} \rfloor + \alpha)) = H^i(W, g_* \mathcal{O}_{\widehat{X}}(mK_{\widehat{X}} - \lfloor \frac{B_m}{bc} \rfloor + \alpha)) = 0$$

for all $\alpha \in \mathrm{Pic}^0(\widehat{X})$ and all $i > 0$. It follows from $\sigma_* \mathcal{O}_{\widehat{X}}(mK_{\widehat{X}/\widetilde{X}}) = \mathcal{O}_{\widetilde{X}}$ that

$$\mathcal{J}_m := \sigma_* \mathcal{O}_{\widehat{X}}(mK_{\widehat{X}/\widetilde{X}} - \lfloor \frac{B_m}{bc} \rfloor)$$

is an ideal sheaf of $\mathcal{O}_{\widetilde{X}}$. One sees that $\mathcal{J}_m \supset \mathcal{I}_{m,\alpha}$, $\mathcal{J}_m|_H \supset \mathcal{I}_{m,H}$ and

$$H^i(A, \nu_* \widetilde{f}_* (\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}} + \alpha) \otimes \mathcal{J}_m)) = 0$$

for all $\alpha \in \mathrm{Pic}^0(\widetilde{X})$ and all $i > 0$, where $\mathcal{I}_{m,\alpha}$ (resp. $\mathcal{I}_{m,H}$) is the base ideal of $|mK_{\widetilde{X}} + \alpha|$ (resp. $|mK_H|$). In particular, $\nu_* \widetilde{f}_* (\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}}) \otimes \mathcal{J}_m)$ is IT^0.

Since

$$\pi_* \mathcal{O}_{\widetilde{X}}(6K_{\widetilde{X}}) = \mathcal{O}_{\widetilde{Y}}(6K_{\widetilde{Y}} + 6L) \oplus \mathcal{O}_{\widetilde{Y}}(6K_{\widetilde{Y}} + 5L),$$

we conclude that there exists ideal sheaves $\mathcal{J}_6^1$ and $\mathcal{J}_6^2$ on $\widetilde{Y}$ such that

$$\pi_* (\mathcal{O}_{\widetilde{X}}(6K_{\widetilde{X}}) \otimes \mathcal{J}_6) = (\mathcal{O}_{\widetilde{Y}}(6K_{\widetilde{Y}} + 6L) \otimes \mathcal{J}_6^1) \oplus (\mathcal{O}_{\widetilde{Y}}(6K_{\widetilde{Y}} + 5L) \otimes \mathcal{J}_6^2).$$

Hence $\nu_* p_* (\mathcal{O}_{\widetilde{Y}}(6K_{\widetilde{Y}} + 5L) \otimes \mathcal{J}_6^2)$ is IT^0.

On the other hand, we know that

$$\begin{aligned} H^0(H, \mathcal{O}_H(6K_H)) &= H^0(H, \mathcal{O}_H(6K_H) \otimes \mathcal{J}_6|_H) \\ &= H^0(F, \mathcal{O}_F(6K_F + 6L|_F) \otimes \mathcal{J}_6^1|_F) \\ &\quad \oplus H^0(F, \mathcal{O}_F(6K_F + 5L|_F) \otimes \mathcal{J}_6^2|_F). \end{aligned}$$

This implies

$$H^0(F, \mathcal{O}_F(6K_F + 6L|_F)) = H^0(F, \mathcal{O}_F(6K_F + 6L|_F) \otimes \mathcal{J}_6^1|_F)$$

and

$$H^0(F, \mathcal{O}_F(6K_F + 5L|_F)) = H^0(F, \mathcal{O}_F(6K_F + 5L|_F) \otimes \mathcal{J}_6^2|_F).$$

Since $|6K_H|$ is birational for a surface H, we obtain $H^0(6K_F + 5L|_F) \neq 0$ by Lemma 2.2. Thus we have

$$H^0(F, \mathcal{O}_F(6K_F + 5L|_F) \otimes \mathcal{J}_6^2|_F) = H^0(F, \mathcal{O}_F(6K_{\widetilde{Y}} + 5L) \otimes \mathcal{J}_6^2|_F) \neq 0.$$

This implies

$$\nu_* p_* (\mathcal{O}_{\widetilde{Y}}(6K_{\widetilde{Y}} + 5L) \otimes \mathcal{J}_6^2)$$

is nonzero. Therefore we conclude that

$$\nu_* p_* (\mathcal{O}_{\widetilde{Y}}(6K_{\widetilde{Y}} + 5L) \otimes \mathcal{J}_6^2)$$

is a nonzero IT^0 sheaf on A.

Hence for any $\alpha \in \mathrm{Pic}^0(A)$, $H^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(6K_{\widetilde{Y}} + 5L + \alpha)) \neq 0$. By [5, Corollary 2.4], one sees that $\varphi_{|6K_{\widetilde{X}} + \alpha|}$ separates two general points on two distinct general fibers of π. Therefore $\varphi_{|6K_{\widetilde{X}} + \alpha|}$ is birational by Lemma 2.2. □

Now we prove the following theorem which is slightly more general than Theorem 1.2.

Theorem 3.2. $|5K_X + \alpha|$ *is birational for any* $\alpha \in \mathrm{Pic}^0(X)$, *if* $\dim V^0(\omega_X) > 0$.

Proof. We take $B \in |bc(m-1)K_X - f^*bM|$ as in Lemma 2.4 and

$$L_m := (m-1)K_X - \lfloor \frac{B}{bc} \rfloor \equiv \frac{1}{c} f^*M + \{\frac{B}{bc}\},$$

$m \geq 2$. One sees that

$$H^0((K_X + L_m)|_S) \cong H^0(mK_S) \text{ and } H^0(K_X + L_m + \alpha) \cong H^0(mK_X + \alpha),$$

for all $\alpha \in \mathrm{Pic}^0(X)$. By [9, Theorem 10.15], we have

$$H^i(A, a_*\mathcal{O}_X(K_X + L_m) \otimes \alpha) = 0,$$

for all $\alpha \in \mathrm{Pic}^0(A)$ and all $i > 0$. In particular, $a_*\mathcal{O}_X(K_X + L_m)$ is IT^0.

Step 1. *Let $x \in X$ be a general point. Then $K_X + L_m$ has no essential base point at x.*

Let $F = X_{a(x)}$. By x being general, we mean that

$$a_*\mathcal{O}_X(K_X + L_m) \otimes \mathbb{C}(a(x)) \cong H^0((K_X + L_m)|_F) \cong H^0(mK_F),$$

F is smooth and x is not a base point of $|(K_X + L_m)|_F|$. Hence pushing forward the standard exact sequence

$$0 \to \mathcal{I}_x(K_X + L_m) \to \mathcal{O}_X(K_X + L_m) \to \mathcal{O}_x \to 0$$

to A, we obtain

$$0 \to a_*(\mathcal{I}_x(K_X + L_m)) \to a_*\mathcal{O}_X(K_X + L_m) \to \mathcal{O}_{a(x)} \to 0.$$

Since $a_*\mathcal{O}_X(K_X + L_m)$ is IT^0, by Lemma 2.6, $a_*\mathcal{O}_X(K_X + L_m)$ has no essential base point at $a(x)$. It follows that $K_X + L_m$ has no essential base point at x.

Step 2. *Let $y \in X$ be a general point. Then $a_*(\mathcal{I}_y(K_X + L_3))$ has no essential base point at any point of Z.*

Because of Lemma 2.6, we want to show that $a_*(\mathcal{I}_y(K_X + L_3))$ is M-regular. As in Step 1, we have the exact sequence

$$0 \to a_*(\mathcal{I}_y(K_X + L_3)) \to a_*\mathcal{O}_X(K_X + L_3) \to \mathcal{O}_{a(y)} \to 0.$$

Considering the long exact sequence obtained from the above short exact sequence, we conclude that $H^i(a_*(\mathcal{I}_y(K_X + L_3)) \otimes \alpha) = 0$ for all $\alpha \in \mathrm{Pic}^0(A)$ and $i \geq 2$.

We now assume that $a_*(\mathcal{I}_y(K_X + L_3))$ is not M-regular. Then by definition of M-regular, we know that

$$\operatorname{codim} V^1(a_*(\mathcal{I}_y(K_X + L_3))) \leq 1.$$

Hence y is a base point of $|K_X + L_3 + \alpha|$ for any $\alpha \in V^1(a_*(\mathcal{I}_y(K_X + L_3)))$. By y being general, we know that y is also a base point of $|3K_X + \alpha|$ for any $\alpha \in V^1(a_*(\mathcal{I}_y(K_X + L_3)))$.

Let C_1 be a component of $V^0(\omega_X)$ with positive dimension, by [6, Theorem 0.1], we can write $C_1 = \alpha_1 + T_1$, where T_1 is a subtorus of $\widehat{A} := \mathrm{Pic}^0(X)$ and α_1 is a point of $\widehat{A}$. Since $y \in X$ is a general point, let $U_1 \subset T_1$ be a non-empty open subset such that y is not a base point of $|K_X + \beta|$, for any $\beta \in \alpha_1 + U_1$.

By considering the map

$$H^0(K_X + \alpha) \otimes H^0(2K_X + \beta) \to H^0(3K_X + \alpha + \beta),$$

we conclude that y is a base point of $|2K_X - \alpha_1 + \beta|$ for any $\beta \in V^1(a_*(\mathcal{I}_y(K_X + L_3))) - U_1$. By y being general, we know that y is also a base point of $|K_X + L_2 - \alpha_1 + \beta|$ for any $\beta \in V^1(a_*(\mathcal{I}_y(K_X + L_3))) - U_1$.

Let D be a component of $V^1(a_*(\mathcal{I}_y(K_X+L_3)))$ such that D is an effective irreducible divisor on $\widehat{A}$. Take a point $\gamma \in D$ and a point $\beta_1 \in T_1$. We can assume that y is not a base point of $|3K_X+\beta_1+\gamma|$. Thus

$$\beta_1 \notin V^1(a_*(\mathcal{I}_y(K_X+L_3))) - \gamma.$$

This implies that $\beta_1 \notin D-\gamma$. We conclude that $T_1 \nsubseteq (D-\gamma)$ and $0 \in T_1 \cap (D-\gamma)$. Let $p : \widehat{A} \to \widehat{A}/T_1$ be the quotient map. Since

$$\dim p(D-\gamma) = \dim(D-\gamma) - \dim T_1 \cap (D-\gamma) = \dim(\widehat{A}/T_1),$$

we know that $p(D-\gamma) = \widehat{A}/T_1$. This implies that $D-\gamma-T_1 = \widehat{A}$, i.e., $D-T_1 = \widehat{A}$. Thus $V^1(a_*(\mathcal{I}_y(K_X+L_3))) - U_1$ contains a non-empty open subset of $\mathrm{Pic}^0(X)$. This contradicts that K_X+L_3 has no essential base point at y.

Step 3. *If $|3K_S|$ is birational, then $|5K_X+\alpha|$ induces a birational map for any $\alpha \in \mathrm{Pic}^0(X)$.*

We need the following:

Claim 3.3. *Let $x_1, x_2 \in X$ be general points. Then $|K_X+L_3+\alpha|$ separates x_1, x_2 for general $\alpha \in \mathrm{Pic}^0(X)$.*

We follow the idea in the proof of [5, Corollary 2.4]. If x_1 and x_2 are on a general F of a. Since $H^0((K_X+L_3)|_F) \cong H^0(3K_F)$, we know that $|(K_X+L_3)|_F|$ separates x_1, x_2. This implies $a_*(\mathcal{I}_{x_1,x_2}(K_X+L_3)) \neq a_*(\mathcal{I}_{x_1}(K_X+L_3))$. Hence we have an exact sequence

$$0 \to a_*(\mathcal{I}_{x_1,x_2}(K_X+L_3)) \to a_*(\mathcal{I}_{x_1}(K_X+L_3)) \to \mathcal{O}_{a(x_2)} \to 0.$$

If $a(x_1) \neq a(x_2)$, we obtain $a_*(\mathcal{I}_{x_1}(K_X+L_3)) \otimes \mathbb{C}(a(x_2)) \cong a_*(K_X+L_3) \otimes \mathbb{C}(a(x_2))$. Thus we still obtain

$$0 \to a_*(\mathcal{I}_{x_1,x_2}(K_X+L_3)) \to a_*(\mathcal{I}_{x_1}(K_X+L_3)) \to \mathcal{O}_{a(x_2)} \to 0.$$

By Step 2, $a_*(\mathcal{I}_{x_1}(K_X+L_3))$ has no essential base point at $a(x_2)$. Thus for general $\alpha \in \mathrm{Pic}^0(X)$

$$h^0(a_*(\mathcal{I}_{x_1,x_2}(K_X+L_3)) \otimes \alpha) = h^0(a_*(\mathcal{I}_{x_1}(K_X+L_3)) \otimes \alpha) - 1.$$

Therefore for general $\alpha \in \mathrm{Pic}^0(X)$

$$h^0(\mathcal{I}_{x_1,x_2}(K_X+L_3) \otimes \alpha) = h^0(\mathcal{I}_{x_1}(K_X+L_3) \otimes \alpha) - 1 = h^0(\mathcal{O}_X(K_X+L_3) \otimes \alpha) - 2$$

and the Claim follows.

For a fixed $\alpha \in \mathrm{Pic}^0(X)$, we choose general $\beta \in \mathrm{Pic}^0(X)$ and consider the map

$$H^0(2K_X+\alpha-\beta) \otimes H^0(3K_X+\beta) \to H^0(5K_X+\alpha).$$

We conclude that $|5K_X+\alpha|$ induces a birational map for any fixed $\alpha \in \mathrm{Pic}^0(X)$.

Step 4. *If $|3K_S|$ is not birational, then $|5K_X+\alpha|$ induces a birational map for any $\alpha \in \mathrm{Pic}^0(X)$.*

Since $|3K_S|$ is not birational, it is well known that S satisfies $(K^2_{S_0}, p_g) = (1,2)$ or $(2,3)$, here S_0 is the minimal model of S. Thus $\varphi_{|3K_S|}$ is a generically double covering onto its image (cf. [1]). The natural morphism $f^*f_*\omega_X^{\otimes 3} \to \omega_X^{\otimes 3}$ define a rational map $g : X \dashrightarrow Y \subset \mathbb{P}(f_*\omega_X^{\otimes 3})$ over W, where Y is the closure of the image.

As the construction in the proof of Theorem 3.1, we have the commutative diagram among smooth projective varieties

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\pi"] \arrow[dr, "\widetilde{f}"'] & \widetilde{Y} \arrow[d, "p"] \\ & W, \end{tikzcd}$$

here the double cover π is a birational modification of g. The arguments in Step 3 show that $\varphi_{|5K_{\widetilde{X}}+\alpha|}$ separates two general points on two distinct general fibers of π for any $\alpha \in \mathrm{Pic}^0(\widetilde{X})$. By the same method in the proof of Theorem 3.1, we can obtain Step 4. We leave the details to the interested reader. $\square$

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